30[34E15, 35B25, 73U05, 78A30, 80A25, 92C20]—*The boundary function method for singular perturbation problems*, by Adelaida B. Vasil'eva, Valentin F. Butuzov, and Leonid V. Kalachev, SIAM Studies in Applied Mathematics, Vol. 14, SIAM, Philadelphia, PA, 1995, xiv+221 pp., 26 cm, $58.50

A. B. Vasil'eva and V. F. Butuzov wrote three books, Vasil'eva and Butuzov [3, 4, 5], on singularly perturbed equations. The book under review is an updated and revised translation of the 1990 book with a new set of exercises, more illustrative examples, and other changes. For example, §17 on the acoustic oscillations in a medium with small viscosity in the 1990 book was replaced by §4.4 on the relaxation waves in the FitzHugh-Nagumo system; §3.4.5 with an example of nonisothermal chemical reaction and §4.3.1 on a one-dimensional model of semiconductor devices were not available in the 1990 book. Also, the references have increased from 121 items to 167 items.

The term *boundary function method*, appearing in the title of the book, may be vague to some people working in the area of singular perturbations. As stated in the Preface, it is also known as the *method of boundary layer correction*, which is a basic technique in seeking an asymptotic expansion, uniformly valid on the given domain, for a singularly perturbed problem. The core of this book reflects a cumulative work of investigating singularly perturbed systems by using this asymptotic technique at the Department of Physics of the Moscow State University since the pioneering paper [1] of A. N. Tikhonov; see Vasil'eva [2]. The main attention of this book is devoted to algorithms for constructing the asymptotic approximations of the solutions of problems motivated by applications in physics, engineering, chemistry, and biology. Justification of the asymptotic correctness of asymptotic solutions is given only for problems whose proofs are short. Otherwise outlines of the proofs are presented or references are mentioned where proofs can be found. The book is intended to be used in courses on asymptotic methods and applied mathematics at the undergraduate or graduate level, as well as for self-study by applied scientists who use asymptotic methods in their work.

There are four chapters. Chapter 1 presents basic ideas of regular and singular perturbations, asymptotic approximations, and asymptotic series along with examples to illustrate the notion of initial and boundary layers. Chapter 2 treats singularly perturbed differential equations, most of which are excerpted from [3, 4]. It begins with the statement of the Tikhonov theorem on the convergence of the solution of the singularly perturbed system of first-order differential equations $\mu \mathbf{z}'(t) = F(\mathbf{z}, \mathbf{y}, t), \mathbf{y}'(t) = f(\mathbf{z}, \mathbf{y}, t)$ defined in $t \in (0, T)$, subject to initial conditions, to a reduced solution as $\mu \downarrow 0$. An asymptotic algorithm is then developed to construct a uniform approximation involving initial layer functions with a remainder of order μ^{n+1}, as proved by Vasil'eva. For a corresponding boundary value problem with two components in $\mathbf{z}$ and a scalar function y defined in $t \in (0, 1)$, subject to initial conditions of y, z_1 at $t = 0$ and a terminal condition of z_2 at $t = 1$, a uniform solution with boundary layer functions at $t = 0$, $t = 1$ is constructed with a remainder of order μ^{n+1}. The next problem is a system with a small nonlinearity, $\mu \mathbf{x}'(t) = A(t)\mathbf{x} + \mu f(\mathbf{x}, t, \mu)$, defined in $t \in (0, T)$, subject to an initial condition. A uniform solution containing initial layer functions is constructed with a remainder of order μ^{n+1}. Attention is also paid to the case of using μ^2 to replace μ as the coefficient of $\mathbf{x}'(t)$, together with an example from chemical kinetics. The equation

$\mu^2 z''(t) = F(z,t)$ defined in $t \in (0,1)$, subject to homogeneous Dirichlet boundary conditions, is studied for shock layer behavior as $\mu \downarrow 0$. The chapter concludes with the construction, in three cases, of the asymptotic expansion for the spike-type solution of the system $\mu^2 u''(x) = f(u,v), v''(x) = g(u,v)$ defined in $x \in (0,l)$, subject to homogeneous Neumann boundary conditions, along with a survey of the stability.

Chapter 3 deals with singularly perturbed partial differential equations. The first problem is a selfadjoint elliptic differential equation $\varepsilon^2 \Delta u - k^2(x,y)u = f(x,y,\varepsilon)$ defined in a bounded planar domain, subject to the homogeneous Dirichlet boundary condition. For a domain with smooth boundary, a uniform approximation involving boundary layer functions along the entire boundary is constructed with the remainder of order ε^{n+1}. In the case of the rectangle $(0,a) \times (0,b)$ as domain, corner layer functions are constructed at each corner in order to obtain a uniform solution. An equation of the form $\varepsilon^2 \Delta u - \varepsilon^\alpha A(x,y)u_y - k^2(x,y)u = f(x,y,\varepsilon)$ defined in the rectangle is also studied. The second type of equation is a system of parabolic equations $\varepsilon^2\{\mathbf{u}_t - a(x,t)\mathbf{u}_{xx}\} = f(\mathbf{u},x,t,\varepsilon)$ defined in $x \in (0,1)$, $t \in (0,T)$, subject to an initial condition and homogeneous Neumann boundary conditions at $x = 0$, $x = 1$. An asymptotic solution containing initial layer functions, boundary layer functions at $x = 0$, $x = 1$, and corner layer functions at points $(0,0)$, $(1,0)$ is constructed with the remainder of order ε^{n+1}. The third type of equation is given by $\varepsilon u_t + b(x)u_x - \varepsilon^2 a(x)u_{xx} = f(u,x,t,\varepsilon)$ defined in $x \in (0,1)$, $t \in (0,T)$, subject to an initial condition and homogeneous Neumann boundary conditions at $x = 0$, $x = 1$. Owing to an incompatibility between initial and boundary data at the inflow corner $(0,0)$ angular layer functions are constructed to obtain a uniform solution with a remainder of order ε^2. This smoothing procedure is also applied to other parabolic problems. The next problem is a system of elliptic equations $\varepsilon^2 \Delta \mathbf{u} = A(x,y)\mathbf{u} + \varepsilon^2 f(\mathbf{u},x,y,\varepsilon)$ defined in a bounded planar domain, subject to Dirichlet boundary conditions. For a domain having smooth boundary, a uniform solution having boundary layer functions along the entire boundary is constructed with a remainder of order ε^{n+1}. The method is also applied to a system of parabolic equations and an example from nonisothermal chemical reaction. The fifth type of equation is of the form $u_t + s(x,t)u + \varepsilon F(u,x,t,\varepsilon) + f(x,t) = \varepsilon^2 u_{xx}$ defined in $x \in (0,l)$, $t \in (0,\infty)$, subject to homogeneous Dirichlet boundary conditions at $x = 0$, $x = l$ and a periodic condition in time with the period 2π. An approximation with boundary layer functions at $x = 0$, $x = l$ is derived with a remainder of order ε^{n+1}. Several variants of this problem are also studied. The last class of problems includes a first-order hyperbolic equation of the form $\varepsilon\{u_t + \Gamma(x,t)u_x\} = a(x,t)u + f(x,t)$ together with a hyperbolic system of two first-order differential equations and telegraphic equations.

Chapter 4 provides asymptotic solutions for four applied problems. The combustion process with a first-order autocatalytic reaction consists of a system of two time-dependent reaction diffusion equations in the dimensionless form $\varepsilon\theta_t - a\theta_{xx} = (v_0 + v)(1-v)\exp(\theta), \varepsilon v_t - bv_{xx} = \varepsilon(v_0 + v)(1-v)\exp(\theta)$ for $x \in (0,1), t \in (0,T)$, subject to the homogeneous initial conditions, the homogeneous Dirichlet boundary conditions for θ at $x = 0$, $x = 1$, and the homogeneous Neumann boundary conditions for v at $x = 0$, $x = 1$. A uniform solution involving initial layer functions is given with a remainder of order ε^{n+1}. For a heat conduction process in a thin rod, one derives the parabolic reaction diffusion equation of the form $\varepsilon^2 u_t - a(x)\{\varepsilon^2 u_{xx} + u_{yy}\} = \varepsilon^2 f(u, x, t)$ defined in the domain $x \in (0,1)$, $y \in (0,1)$, $t \in (0,T)$, subject to an initial condition, the Dirichlet boundary conditions at $x = 0$, $x = 1$, and the Robin boundary conditions at $y = 0$, $y = 1$. A uniform approximation containing an initial layer function, boundary layer functions at $y = 0$, $y = 1$, and corner layer functions is constructed with the remainder of order ε. A one-dimensional semiconductor device is described as a system $\mu^2 E'(x) = p - n + N(x)$, $n'(x) = -nE + I_n$, $p'(x) = pE - I_p$ defined in $x \in (0,1)$, subject to the boundary conditions $n(0) = p(0)$, $n(1) = 1$, $p(1) = 0$. A uniform approximation of the zeroth order having boundary layer functions at $x = 0$, $x = 1$ is given. Its extension to a two-dimension model is provided as well. The fourth problem is a model for the propagation of excitation in a nerve axon, which can be formulated as the FitzHugh-Nagumo system $\varepsilon u_t - \varepsilon^2 u_{xx} = u(a-u)(u-1) - v + I$, $v_t = bv - \gamma v$. A uniform approximation for the relaxation wave solution (or periodic traveling wave solution) of this system involving shock layer functions is constructed with a remainder of order ε. This chapter is furnished with references to problems arising from optimal control, rigid body dynamics under electro- and hydrodynamic influences, a variety of semiconductor structures, molecular aerodynamics, theory of alloys, theory of neutrons, theory of epidemics, heat and mass transfer in a two-component medium, and acoustic oscillations in a medium with small viscosity.

Typesetting of mathematical equations in the book is better than that of its 1990 Russian counterpart. Some obvious typographical errors remain, however. Compared to other books in the area of singular perturbations, the strength of this book is to blend nicely systems of semilinear differential equations, linear and semilinear partial differential equations, as well as some applied problems. Experts in the area of singular perturbations and related fields should find part, if not all, of the book extremely useful. It can also serve as a textbook for a two-semester graduate course in singular perturbations.

References

1. A. N. Tikhonov, *On the dependence of the solutions of differential equations on a small parameter*, Mat. Sbornik N. S. **22** (64) (1948), 193–204. (Russian) MR **9**:588
2. A. B. Vasil′eva, *On the development of singular perturbation theory at Moscow State University and elsewhere*, SIAM Rev. **36** (1994), 440–452. MR **95h**:34079
3. A. B. Vasil′eva and V. F. Butuzov, *Asymptotic expansions of the solutions of singularly perturbed equations*, Nauka, Moscow, 1973. (Russian) MR **57**:16876
4. ———, *Singularly perturbed equations in critical case*, Moskov. Gos. Univ., Moscow, 1978 (Russian). An English translation by F. A. Howes as Math. Res. Center TSR 2039, University of Wisconsin, Madison, Wisconsin, 1980. MR **80b**:34070
5. ———, *Asymptotic methods in the theory of singular perturbation*, Vischaja Shkola, Moscow, 1990. (Russian) MR **92i**:34072

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